Robust Linear Quadratic Gaussian Control for Flexible Structures

P. A. Blelloch*
Structural Dynamics Research Corporation, San Diego, California
and
D. L. Mingori†
University of California, Los Angeles, California

A procedure is developed to deal with performance and robustness issues in the design of multiple-input multiple-output compensators for lightly damped flexible structures. The procedure is based upon representing errors in the plant design model as structured uncertainties, and applying a modified linear quadratic Gaussian design method. The cost function in the regulator problem and the process noise model in the estimator problem are varied in a manner that reflects specific parameter uncertainty including frequency, damping, or modal displacement errors. Numerical examples dealing with the control of a large flexible space antenna with uncertain frequencies demonstrate the application of the method.

I. Introduction

ROBUST compensator design for flexible structures involves maintaining closed-loop stability in the face of several types of model errors. Two of the most important are unmodeled dynamics and parameter errors in the modeled dynamics. The present work focuses on the latter issue. Once a reduced-order model of a flexible plant has been determined, usually by some form of modal truncation, robustness with respect to parameter errors may be addressed within the context of a finite-dimensional problem. The approach taken in this paper, therefore, is based on a fixed-order finite-dimensional model of the flexible structure. The method is applicable to any plant parameter uncertainty, but since the closed-loop stability of flexible structures is especially sensitive to errors in modal frequencies, the present work concentrates on robustness with respect to frequency errors.

The approach described here was motivated by the μ-synthesis method proposed by Doyle,[1-3] which guarantees stability of a closed-loop design for all systems whose dynamics remain within prescribed bounds relative to the nominal design model. The computational procedure required for μ-synthesis is substantially more complex than standard linear quadratic Gaussian (LQG) methods and is not directly applicable to real parameter variations. In contrast, the approach presented here does not guarantee stability for modeling errors within prescribed bounds, but it does permit a controlled tradeoff between performance and robustness, while using standard, well-tested numerical methods.

A very similar approach to that presented here is developed in Ref. 4, where the authors suggest an "internal feedback loop" to represent parameter uncertainty. This is identical to the structured uncertainty presented in the next section. The authors of Ref. 4 derive the important result that if certain similarity and minimum-phase conditions are satisfied, the closed-loop poles can be made asymptotically insensitive to parameter uncertainties by allowing particular weighting matrices in the linear quadratic regulator (LQR) problem and covariance matrices in the Kalman-Bucy filter (KBF) problem to approach infinity. Because of the way these matrices are chosen, the regulator and filter gains will approach infinity as well. This procedure is also used in the present paper, but here it is motivated as an L_2 minimization of signals in a structured uncertainty model. We apply the method to the special but important case of controlling a flexible structure with several uncertain frequencies and a small number of actuators and measurements. For this problem, the similarity conditions of Ref. 4 are not satisfied, but the results show that the method can still improve robustness.

The organization of this paper is as follows. Section II presents the structured uncertainty representation of parameter errors for a flexible structure.[1-3,5-7] It also develops the robust LQG method and discusses some relevant results from Ref. 4. Section III gives a more traditional LQG interpretation of the method, where the parameter uncertainties are represented by white noise processes. This interpretation is briefly compared to the more rigorous, maximum entropy stochastic control design method of Hyland and Bernstein.[8] Section IV presents an example based on a wrap-rib antenna.[9-11] Both the regulator problem and the estimator problem are presented. Finally, we make some conclusions in Sec. V. The procedure is also discussed in greater detail in Refs. 12 and 13.

II. Development of Robust LQG Control

The problem treated here is to choose a control design that provides both adequate performance and robust stability for a set of plants, rather than a single "nominal" plant. In particular, since closed-loop stability for lightly damped flexible structures appears to be especially sensitive to frequency variations, we will consider this effect. The method, however, can be extended to include other parameter uncertainties such as damping ratios or mode shapes, and is not limited to the case of lightly damped flexible structures.

A number of methods for describing plant uncertainty have been used in analyzing the robust stability of a system. One description that takes into account a broad class of model uncertainties is the structured uncertainty representation shown in Fig. 1. This representation permits one to make different assumptions concerning the uncertain, norm-bounded matrix Δ (e.g., block diagonal, real, etc.) in order to describe different types of modeling errors. For example, one special case is the unstructured uncertainty where Δ is a complex, norm-

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*Senior Project Engineer. Member AIAA.

†Professor, Mechanical, Aerospace and Nuclear Engineering Department. Associate Fellow AIAA.

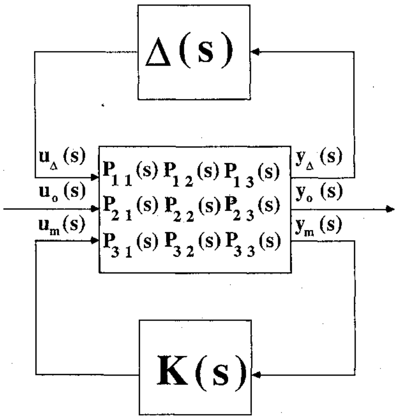

Fig. 1 Structured uncertainty representation.

bounded matrix with no structure imposed on it.[14,15] The unstructured uncertainty is useful when all model uncertainties can be nonconservatively lumped into a single matrix Δ. However, the use of unstructured uncertainties can be shown to lead to extremely conservative designs for lightly damped flexible structures with uncertain frequencies. Other cases that have been studied include the case of a block diagonal Δ matrix, with norm-bounded blocks, and real diagonal Δ matrices,[5,6] applicable to the situation of parameter uncertainties such as the special case of uncertain natural frequencies in a flexible structure.

A discussion of the structured uncertainty representation for the special case of uncertain real model parameters will follow. Consider the case where M real parameter uncertainties exist in the model system matrix A. In the absence of external inputs, the uncertain model can then be written as

$$\dot{x} = Ax + \left[\sum_{i=1}^{M} c_i A_i\right] x \tag{1}$$

where each c_i is an uncertain parameter that can vary between $+1$ and -1, and A_i describes the effect of the ith parameter uncertainty on the system matrix A. Now let k_i be the rank of the A_i and factor A_i as $\alpha_i \beta_i$, where α_i is an $n \times k_i$ matrix, and β_i is a $k_i \times n$ matrix (the factorization is not unique). A state-space representation of $P_{11}(s)$ in Fig. 1 can then be written as[5,6]

$$\begin{Bmatrix} \dot{x} \\ y_\Delta \end{Bmatrix} = \begin{bmatrix} A & \alpha \\ \beta & 0 \end{bmatrix} \begin{Bmatrix} x \\ u_\Delta \end{Bmatrix} \tag{2}$$

where $\alpha = [\alpha_1 \, \alpha_2, \, ..., \, \alpha_m]$, $\beta = [\beta_1^T \, \beta_2^T, \, ..., \, \beta_m^T]^T$, and u_Δ and y_Δ are the inputs and outputs of $P_{11}(s)$, respectively. In this case, Δ will have the following structure:

$$\Delta = \begin{bmatrix} c_1 I_{k1} & 0 & . & . & 0 \\ 0 & c_2 I_{k2} & . & . & . \\ . & . & . & . & . \\ . & . & . & . & 0 \\ 0 & . & . & 0 & c_m I_{km} \end{bmatrix} \tag{3}$$

The structure imposed on Δ in this case is that Δ remain diagonal with m real parameters as shown previously. To see that this is a correct representation for $P_{11}(s)$, let $u_\Delta = \Delta y_\Delta$, and substitute back into Eq. (2) to get

$$\dot{x} = Ax + \left[\sum_{i=1}^{M} c_i \alpha_i \beta_i\right] x = \left[A + \sum_{i=1}^{M} c_i A_i\right] x \tag{4}$$

which is the original uncertainty model. Now append inputs and outputs, a noise model, and a quadratic cost functional to arrive at the standard LQG optimal-control problem with the addition of the uncertain dynamics

$$\dot{x} = Ax + \left[\sum_{i=1}^{M} c_i A_i\right] x + B u_m + D w \tag{5a}$$

$$y_m = Cx + v \tag{5b}$$

$$J = E\left[\int_0^\infty (x^T Q x + u^T R u) \, \mathrm{d}t\right] \tag{5c}$$

where u_m and y_m are control inputs and measurements, respectively, and w and v are Gaussian white noise processes with spectral density matrices $\hat{Q}$ and $\hat{R}$, respectively. The optimal-control problem consists of minimizing $\|Q^{1/2} + R^{1/2} u\|_2 = \int_0^\infty (x^T Q x + u^T R u) \, \mathrm{d}t$ for white noise inputs. The LQG problem can be represented as a special case of the structured uncertainty representation illustrated in Fig. 1. For an LQG problem with model uncertainty, a state-space representation of the dynamic operator $P(s)$ will have the following form:

$$\begin{Bmatrix} \dot{x} \\ y_\Delta \\ y_0 \\ y_m \end{Bmatrix} = \begin{bmatrix} A & \alpha & D\hat{Q}^{1/2} & B \\ \beta & 0 & 0 & 0 \\ Q^{1/2} & 0 & 0 & R^{1/2} \\ C & 0 & \hat{R}^{1/2} & 0 \end{bmatrix} \begin{Bmatrix} x \\ u_\Delta \\ u_0 \\ u_m \end{Bmatrix} \tag{6}$$

In this case, u_Δ and y_Δ are the inputs and outputs representing model uncertainty, y_0 is the output to be minimized, u_0 is the unit covariance white noise process (i.e., $u_0 = [[\hat{Q}^{-1/2} w]^T [\hat{R}^{-1/2} v]^T]^T$), y_m is the vector of control measurements, and u_m is the vector of control inputs. If we ignore the uncertainty model, minimizing the 2-norm of the dynamic operator from u_0 to y_0 reduces to the standard LQG optimal-control problem. When we include the uncertainty model, however, the problem becomes more complex, since u_Δ is not a white noise process. In the case where Δ is an unstructured uncertainty and u_Δ and y_Δ represent the only inputs and outputs, the optimal-control problem can be solved by minimizing the L_∞ norm ($\|G(s)\|_\infty = \max_\omega \bar{\sigma}[G(j\omega)]$) of the closed-loop transfer-function matrix from u_Δ to y_Δ (H^∞ control). The next most general case is where Δ is block diagonal with norm-bounded unstructured blocks and the disturbance inputs u_0 are L_2 bounded functions rather than white noise. In this case the optimal-control problem can be solved approximately by the μ-synthesis method, which is an iterative H^∞ minimization. However, neither case directly applies to the case of real parameter uncertainties where Δ is real and diagonal. Given standard LQG solution tools, however, it is possible to minimize the 2-norm of the closed-loop dynamic operator in Eq. (6). This is equivalent to an LQG problem where the vector u_Δ is appended as an additional noise input, and y_Δ is added to the minimized outputs. A technical error in applying this approach to the uncertainty model is that minimizing the 2-norm is only strictly valid when all inputs are unit covariance white noise processes. For the case under consideration, u_Δ is an L_2 bounded function ($\|u_\Delta\|_2 \le \bar{\sigma}[\Delta] \|y_\Delta\|_2$), but is clearly not a white noise process since it is a function of the dynamic state vector x. The procedure suggested in this paper overlooks this "technical" problem and applies the LQG method directly to the uncertainty model. The modified LQG problem can then be presented as

Given:

$$\dot{x} = Ax + Bu + Dw + \alpha u_\Delta, \qquad E[w(t) w^T(\tau)] = \hat{Q}\delta(t - \tau),$$

$$E[u_\Delta(t) u_\Delta^T(\tau)] = r_1 I \delta(t - \tau)$$

$$y = Cx + v, \qquad E[v(t) v^T(\tau)] = \hat{R}\delta(t - \tau)$$

Minimize:

$$J = E\left[\left\{\int_0^\infty x^T [Q + r_2 \beta^T \beta] x + u^T R u\right\} \mathrm{d}t\right] \tag{7}$$

The preceding method is applicable to linear systems with arbitrary parameter uncertainty in the dynamic system matrix A. To illustrate some the these ideas using a physical system, consider a flexible structure with uncertain natural frequencies, described by the following equations:

$$\ddot{q} + 2Z\Omega\dot{q} + (I + \Delta E)\Omega^2 = Bu + Dw \tag{8a}$$

$$y = [C_1 C_2]\begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} + v \tag{8b}$$

where

q = modal degrees of freedom
Z = diag $\{\zeta_i\}$, ζ_i is the damping ratio in the ith mode
Ω = matrix of nominal frequencies (Ω = diag $\{\omega_i\}$)
u = control forces and torques
w = Gaussian process noise $\{E[w(t)w^T(\tau)] = \hat{Q}\delta(t-\tau) \geq 0\}$
y = measurement outputs
v = measurement noise $\{E[v(t)v^T(\tau)] = \hat{R}\delta(t-\tau) > 0\}$
Δ = diag $\{c_i\}$, $\quad -1 \leq c_i \leq 1$
E = diag $\{\epsilon_i\}$, $\quad (1-\epsilon_i)\omega_i^2 \leq \hat{\omega}_i^2 \leq (1+\epsilon_i)\omega_i^2$

and the following LQR objective function

$$J = \int_0^\infty \{q^T \dot{q}^T\}\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}\begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} + u^T R u \, \mathrm{d}t \tag{9}$$

The c_i are time-invariant, uncertain parameters, while ϵ_i provides a bound on the error in the ith frequency. Note that in this formulation we have lumped the uncertainty into the imaginary part of the eigenvalues, assuming that the real parts remain fixed. A state-space representation for the dynamic operator $P(s)$ can be written using the same format as Eq. (6). Again treating both u_0 and u_Δ as disturbance inputs, and both y_0 and y_Δ as outputs to be minimized, the following LQG problem is suggested:

Given:

$$\ddot{q} + 2Z\Omega\dot{q} + \Omega^2 q = Bu + Dw + \Omega u_\Delta \tag{10a}$$

$$E[w(t)w^T(\tau)] = \hat{Q}\delta(t-\tau), \quad E[u_\Delta(t)u_\Delta^T(\tau)] = r_1 I\delta(t-\tau) \tag{10b}$$

Minimize:

$$J = E\left[\int_0^\infty \{q^T \dot{q}^T\}\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}\begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} + r_2 q^T \Omega^2 q + u^T R u \, \mathrm{d}t\right] \tag{11}$$

The terms Ωu_Δ in Eq. (10a), $r_1 I\delta(t-\tau)$ in Eq. (10b), and $r_2 q^T \Omega^2 q$ in Eq. (11) are added to represent frequency uncertainty. The two parameters r_1 and r_2 can be used to vary the relative importance of these terms. Increasing r_1 increases the emphasis of the uncertainty model in the KBF design, whereas increasing r_2 increases the emphasis of the uncertainty model in the LQR design. Since "optimal" performance might be achieved for the case where $r_1 = r_2 = 0$, the problem then reduces to a simple case of trading performance against robustness. Note, however, that increasing r_1 and r_2 does not necessarily reduce performance. In particular, the term multiplied by r_2 in the cost functional represents the elastic strain energy of the structure. If the performance index also included a term that minimized the strain energy, a more robust control system might also achieve higher performance.

As mentioned in the introduction, Ref. 4 uses a different motivation to suggest the same LQG design procedure. Tahk and Speyer show that if certain conditions are satisfied concerning the properties of the error matrix and the number of inputs and measurements, the closed-loop poles can be made asymptotically insensitive to parameter variations as the design parameters r_1 and r_2 are allowed to become arbitrarily large. The problem treated in this paper does not satisfy the conditions of Ref. 4 since the number of varying parameters is greater than either the number of inputs or the number of measurements. Another important result concerns the nonuniqueness involved in factoring A_i as $\alpha_i\beta_i$. In Ref. 4 it is shown that the asymptotic insensitivity is not a function of the choice of factorization. It should be pointed out, however, that the control law for *finite* r_1 and r_2 does depend on the choice of factorization. The choice of factorization is discussed further in the next section.

III. Alternate Intepretation

It is possible to motivate the robust LQG formulation without relying on the structured uncertainty representation. This motivation is in line with traditional LQG control design, where process noise is used to reflect model uncertainty.[16] To provide this motivation, rearrange Eq. (8) so that the uncertainty lies on the right-hand side of the equation

$$\ddot{q} + 2Z\Omega\dot{q} + \Omega^2 q = Bu + Dw - E\Omega^2 q \tag{12}$$

The uncertainty now appears as a disturbance input into the model. Though it is clear that $E\Omega^2 q$ is not a Gaussian white noise process since it is a function of the modal state vector q, it is considered as one in the design procedure used here. This further illustrates the approximation made in solving the structured uncertainty problem using LQG techniques. Suppose $E\Omega^2 q$ is factored to arrive at the following representation:

$$\ddot{q} + 2Z\Omega\dot{q} + \Omega^2 q = Bu + Dw + D'w' \tag{13}$$

where

$$D' = -\Omega, \qquad w' = E\Omega q$$

We treat w' as a white noise process with unit spectral density, and so we add a noise process with spectral density Ω^2 to the Kalman filter formulation. This is identical to the term Ωu_Δ in Eq. (10a). Note that w' is not a unit covariance Gaussian noise process, but is a function of the states. We can, therefore, minimize w' by introducing a term of the form $E^2\Omega^2$ into the LQR cost functional. This is identical to the $q^T\Omega^2 q$ term appearing in Eq. (11). Considering the approach directly from an LQG point of view helps to clarify the assumptions made in treating the structured uncertainty as an LQG problem. These assumptions are not strictly valid since the parameter is treated both as an independent while noise process and as a function of the states. This is the same approximation made in treating a structured uncertainty representation using LQG control design.

As noted in the previous section, the factorization of $E\Omega^2$ is not unique. This issue is treated in greater detail in Ref. 4 where it is shown that the choice of factorization does not affect the asymptotic insensitivity, though it certainly does affect the LQG problem that is solved. In the present case, the factorization was chosen such that the terms in the cost functional and the process noise covariance are equal, and so that the term in the cost functional represents the weighted strain energy of the structure (i.e., $\alpha = \Omega$, $\beta = E\Omega$). An alternate factorization might be $\alpha = I$, $\beta = E\Omega^2$, in which case the artificial process noise would excite each mode equally, while modes would be weighted in proportion to the fourth power of frequency in the cost functional. This would clearly result in a different control law for finite r_1 and r_2, though the same robustness results may hold in the limit. The factorization chosen in this study is physically appealing since it suggests that minimizing the strain energy associated with uncertain modes will result in a more robust control system.

Another approach to robust control design that treats uncertainties as noise processes is the stochastic control design method of Hyland and Bernstein.[8] In this method, the uncer-

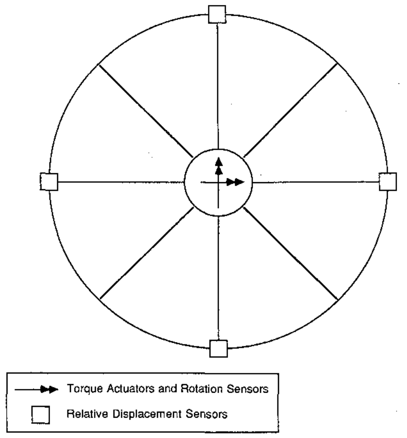

Fig. 2 Wrap-rib antenna model.

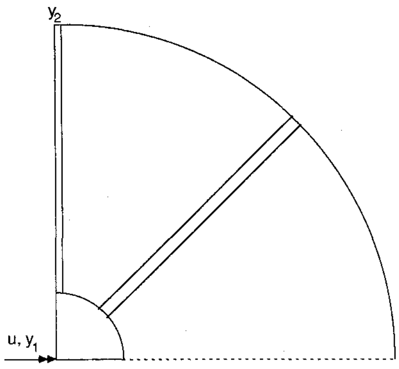

Fig. 3 Quadrant model.

tain parameter term ($E\Omega^2$ rather than $E\Omega^2 q$) is formally treated as a stochastic process. The solution involves solving a coupled set of two Ricatti equations and two Lyapunov equations by an iterative method. Since the state is not included in the noise term ($E\Omega^2$), the issue of treating a function of the state as white noise does not arise in Hyland and Bernstein's approach. It should be noted that the interpretation of uncertainties as a white noise process in the present approach is made only to clarify some issues involved in treating a structured uncertainty model with LQG theory. The motivation for the approach and the results are, therefore, different from those presented in Ref. 8.

IV. Example

In order to demonstrate the method presented here, we have chosen a model of the 180-ft-diam wrap-rib antenna illustrated in Fig. 2.[9,10] The term wrap-rib means that the ribs are initially wrapped around the hub and then released during deployment to achieve the full antenna shape. The control problem consists of actively controlling the out-of-plane motion so as to minimize the overall antenna rms surface error with respect to its nominal position in space. Actuators provide orthogonal torques at the hub center, while measurement of the inertial hub rotation and relative tip deflections are available for feedback. Based on considerations of symmetry, the problem can be reduced to a single quadrant with one torque actuator at the hub, one rotation measurement at the hub, and one relative displacement measurement at the tip of the rib as illustrated in Fig. 3.

The antenna is modeled using a component mode synthesis approach, i.e., mode shapes of the ribs and the mesh sections are found separately, and the "component mode" representations are then combined to form a model of the 90-deg sector. Damping is assumed to be viscoelastic (i.e., the damping matrix is proportional to the stiffness matrix), but we allow for different damping coefficients for the rib and mesh. This implies that damping is added at the component mode stage, and since the damping coefficient for the mesh is assumed to be higher than that for the ribs, the damping matrix, after combining the rib and mesh component mode models and calculating the overall modal representation, will include off-diagonal terms. The final antenna model, in a second-order modal representation, has the following form:

$$\ddot{q} + D\dot{q} + \Omega^2 q = \phi_1^T u, \qquad \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} q \tag{14}$$

where

q = vector of modal coordinates
D = modal damping matrix (including off-diagonal terms)
Ω = diagonal matrix of modal frequencies ($\Omega = \text{diag}\{\omega_i\}$)
ϕ_1 = row vector of mode shape coefficients for hub rotation
ϕ_2 = row vector of mode shape coefficients for relative tip deflection
u = torque applied to hub
y_1 = hub rotation
y_2 = relative tip deflection

The rms surface error is the deformation of the mesh relative to its desired shape and orientation. This error includes a contribution due to rigid body rotation of the antenna and elastic deformation of the mesh surface. Since the antenna is described by a finite-element model, the rms error can be expressed as a summation of the nodal degrees of freedom. The expression for rms error in terms of physical coordinates is then transformed to modal coordinates, resulting in a performance measure of the form $q^T Q_{\text{rms}} q$, which is included in an LQG compensator design. Note that the rms error is a function of the complete modal state vector and not simply a function of plant outputs.

The model used in this study has 63 modes, of which 32 are controllable and observable. The modes are ordered and numbered on the basis of approximate singular values from internal balancing theory rather than on the basis of frequency alone.[17,18] These singular values provide a measure of each mode's participation in the transfer function from plant inputs to plant outputs. (The singular values from internal balancing theory are also known as the Hankel singular values or balanced singular values.) The number of modes included in the design model is then chosen by adding modes one at a time until the compensator design converges.[11,19,20] Based on this method, an 8-mode design model was chosen. The choice of model order is not uncoupled from the control design itself, since higher performance designs will typically require higher-order design models. To clarify the issue of robustness with respect to frequency uncertainty, however, the design model order was kept constant throughout this study. All compensator designs presented here result in identical robustness properties when applied to either the 8-mode design model or the full-order 32-mode model.

The design model consists of one rigid-body mode and seven flexible modes (modes 1–8 in Table 1). For simplicity, the bounds on the uncertainties in modal frequencies are considered to be the same for all seven flexible modes. Thus, E in Eq. (8) becomes ϵI. Since the problem defined here has seven varying parameters and only one input and two outputs, it does not satisfy the similarity conditions of Ref. 4. Nevertheless, we show that robustness with respect to these parameter variations is still improved. Two robustness checks were carried out on each compensator design. The first involved determining the maximum increase and decrease in modal frequencies for which the system remains stable when each nominal frequency ω_{io} is shifted uniformly in the same direction by an

Table 1 Modal data for 32-mode model

Mode no.	Frequency, rad/s	Damping ratio	Singular value	ϕ_1	ϕ_2
1	0.0000	0.000	Infinite	9.3233E−2	0.0000
2	6.9518	0.011	2.2832E−1	−1.2265E−1	2.4336E−1
3	36.5703	0.059	9.4973E−2	−5.1062E−1	6.1749E−1
4	18.9483	0.030	8.4589E−2	−2.8537E−1	1.8319E−1
5	88.7310	0.146	5.5286E−2	−6.5161E−1	5.5838E−1
6	60.3165	0.087	5.3153E−2	−9.7887E−1	1.0835
7	122.5824	0.204	3.2785E−2	−1.1015	9.9704E−1
8	162.2308	0.270	2.1059E−2	−1.1259	1.2321
9	59.6226	0.064	1.2663E−2	−2.7376E−1	2.2423E−1
10	51.8901	0.052	1.0510E−2	−1.9977E−1	2.0086E−1
11	208.8346	0.357	1.0077E−2	−1.0571	9.5021E−1
12	262.7883	0.451	6.0116E−3	−9.7602E−1	1.0861
13	55.3231	0.054	2.9937E−3	−1.1122E−1	1.1487E−1
14	62.1856	0.061	5.2656E−3	−1.7319E−1	1.5371E−1
15	323.8251	0.557	2.9634E−3	−8.9571E−1	7.8892E−1
16	392.3096	0.676	1.9968E−3	−8.3594E−1	9.5116E−1
17	469.4828	0.809	1.3553E−3	−8.7738E−1	7.7879E−1
18	64.8560	0.063	9.8015E−4	7.7735E−2	−6.7832E−2
19	120.6419	0.117	7.1896E−4	1.2184E−1	−1.1314E−1
20	79.1261	0.076	6.3028E−4	−7.3710E−2	7.2487E−2
21	74.9235	0.072	4.3602E−4	−5.7412E−2	5.8941E−2
22	57.5002	0.056	1.7262E−4	−2.5648E−2	3.4725E−2
23	115.5335	0.111	1.5144E−4	5.3449E−2	−4.9332E−2
24	132.4523	0.127	1.0735E−4	5.0889E−2	−4.9705E−2
25	102.9878	0.099	1.0568E−4	−3.7707E−2	4.2910E−2
26	71.1870	0.069	8.7431E−5	−2.3696E−2	2.7136E−2
27	139.2854	0.134	7.2928E−5	4.4394E−2	−4.2185E−2
28	67.8489	0.065	5.9894E−5	2.0735E−2	−1.5010E−2
29	126.2708	0.121	3.5962E−5	2.7592E−2	−2.8840E−2
30	110.9269	0.107	2.6758E−5	2.1996E−2	−1.8513E−2
31	146.8689	0.141	9.3827E−6	1.7430E−2	−1.3913E−2
32	107.0564	0.103	1.7085E−6	−4.2419E−3	7.7825E−3

Table 2 Damping matrix for 8-mode design model

0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	1.463E−1	−5.040E−2	−3.050E−2	−9.605E−2	−6.476E−2	−1.076E−1	−1.102E−1
		4.308	−1.512E−1	−5.052E−1	−4.065E−1	−5.923E−1	−6.045E−1
			1.144	−2.837E−1	−1.794E−1	−3.209E−1	−3.271E−1
				2.585E+1	−6.704E−1	−1.244	−1.264
					1.052E+1	−8.570E−1	−7.363E−1
	SYMMETRIC					4.992E+1	−1.417
							8.924E+1

amount $\rho\omega_{io}$. This rough check offers a simple and computationally efficient measure of the effect of stiffening and softening the structure, but does not consider the situation where frequencies can shift in different directions. The second test mixes the direction of frequency shifts by examining the vertices of the parameter space obtained when the ith natural frequency is allowed to vary by an amount $\pm\rho\omega_{io}$. For m varying frequencies, the resulting parameter space is an m-dimensional hypercube, and so for each value of ρ, 2^m vertices must be checked. For a seven-varying frequencies, this would entail 128 stability checks. Since variations in the last three frequencies of the design model (modes 6–8 in Table 1) were found to have little effect on robust stability, these frequencies were set to $(1-|\rho|)\omega_{io}$. This results in the need to check only 16 points for each percentage frequency variation ρ.

Modal parameters for all 32 modes are listed in Table 1. The singular value column contains the approximate values from internal balancing theory, and the modes are ordered on the basis of these singular values. The damping ratios are based on the diagonal elements of the damping matrix, but the reader should be aware that the damping matrix *does* couple the modal degrees of freedom. The complete damping matrix for the 8-mode design model is listed in Table 2. A diagonal representation should be sufficient for modes 9–32. Note that the first modal damping ratio is 1.1%, while all higher modes are more heavily damped. Although this is higher than is sometimes assumed for lightly damped flexible structures, the assumption of viscoelastic damping is physically realistic, and the purpose of this work is to compare the method suggested here with traditional LQG design techniques. In particular, we will show that a standard LQG/LTR design[14,21] is very sensitive to variation in modal frequencies, even with the higher damping.

Now consider the following hypothetical specifications on the open-loop frequency response from the plant input to the controller output. This is a scalar transfer function for this example.

bandwidth $= \approx 5$ rad/s
loop gain $= 60$ dB at 0.1 rad/s
phase margin $= 60$ deg
gain margin $= 20$ dB

These might be derived from disturbance rejection considerations (where disturbances act at the actuator inputs), or they might be derived from actuator uncertainty considerations. They can be met by a standard LQG/LTR design approach.[14,21] In particular, assume a control problem in the form of Eq. (11). The LQG/LTR approach begins with the selection of Q and R matrices leading to full-state feedback

Table 3 RMS matrix for 8-mode design model

2.166E+4	3.853E+3	−7.545E+3	−3.134E+3	3.389E+3	2.065E+3	1.576E+3	−6.617E+2
	1.619E+4	4.280E+2	−6.330E+3	7.683E+3	1.474E+4	1.530E+3	3.209E+3
		7.372E+4	−7.719E+3	−1.009E+4	1.858E+4	−2.930E+3	−1.336E+3
			1.599E+4	−2.366E+3	−1.941E+4	1.748E+4	−1.844E+2
				1.357E+6	2.692E+2	3.413E+3	−6.515E+2
					2.875E+5	−1.759E+3	−1.280E+3
	SYMMETRIC					3.849E+5	1.975E+2
							9.249E+3

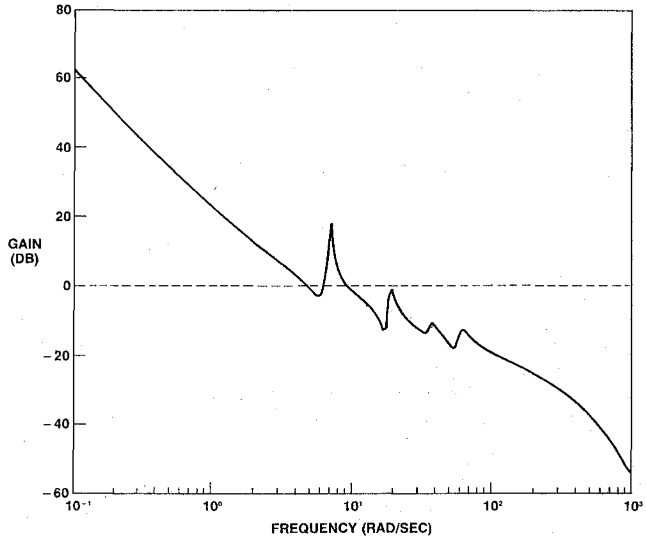

Fig. 4 Input loop shape for standard LQG/LTR control.

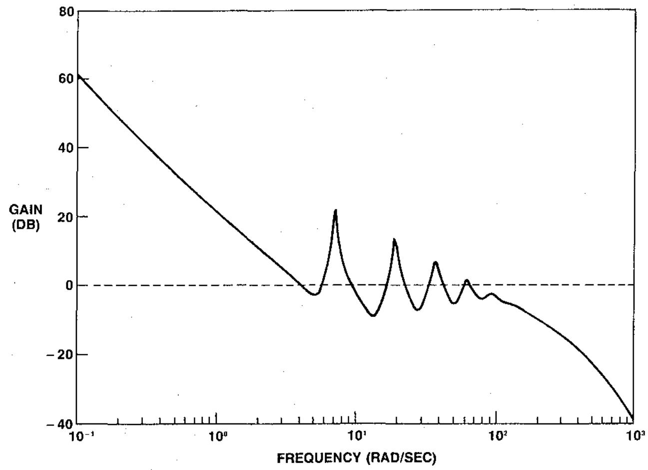

Fig. 5 Input loop shape for modified LQG/LTR control.

gains such that the full-state feedback controller satisfies the design specifications. The parameters r_1 and r_2 are set equal to zero, and the uncertainty model is disregarded. Next, the full-state feedback loop shape is approximated using an observer-based controller. This is accomplished by designing a Kalman-Bucy filter under the assumption that process noise with covariance matrix qI enters the plant at the actuator input. Thus, the covariance matrix for the filter design becomes $\hat{Q} = qBB^T$. The full-state feedback loop shape is recovered by letting q become large.

For this example the full-state feedback design is chosen corresponding to Q and R weighting matrices of

$$Q = Q_{\text{rms}}, \qquad R = 1$$

The matrix Q_{rms} for the eight-mode design model is listed in Table 3. Figure 4 shows the transfer-function gain for a corresponding observer-based design developed using LQG/LTR methods. The process and measurement noise covariance matrices used for this design are

$$\hat{Q} = (1 \times 10^8)BB^T, \qquad \hat{R} = I$$

The gain margin of the LQG/LTR design is 27 dB and the phase margin is 67 deg. Hence, this design meets the design specifications and exhibits good robustness as measured by the traditional measures of gain and phase margin. However, a simple check shows that a uniform increase of 7% in the modal frequencies results in instability of the closed-loop system, indicating that the seemingly robust system is still very sensitive to errors in natural frequency.

Next use the method developed here and take the uncertainty model into account to improve robustness. As a first step add white noise at the uncertainty model input. This is done by increasing the parameter r_1 in Eq. (10b) to a value of 10^4. The resulting loop shape is identical to that illustrated in Fig. 4, but the closed-loop system is now stable for 35% uniform increases in all modal frequencies. It is also stable for 20% decreases and first goes unstable for a 19% increase in the first, second, and fourth modal frequencies coupled with 19% decreases in the other four. Robustness can be further improved by penalizing the output of the uncertainty model.

Fix all other parameters and let $r_2 = 1000$. The loop shape for this case is illustrated in Fig. 5. The gain margin is now reduced to 14dB and the phase margin to 63 deg, but the closed-loop system is stable for 90% uniform increases in frequency along with 50% decreases. The system first goes unstable for a 42% increase in the second and fourth modal frequencies coupled with 32% decreases in the other five. As well as showing an improvement in robustness when applying the method suggested here, these results also indicates that gain and phase margins can provide a very poor measure of robustness for the case of lightly damped flexible structures with uncertain natural frequencies.

It is worthwhile making some qualitative comparisons between the sensitive and robust control designs. The cost functional for the robust design [Eq. (12)] places considerably greater emphasis on the higher modes. This is because the matrix Ω^2 weights each mode in proportion to the square of its natural frequency. The performance measure for this case (weighting rms error), on the other hand, places relatively higher weighting on the rigid-body mode. The robust design, therefore, results in closed-loop regulator poles that lie further to the left with increasing frequency. The sensitive design, on the other hand, attempts to push the closed-loop poles corresponding to the rigid-body mode further to the left than those corresponding to the flexible modes. This is also demonstrated by the loop gains (Figs. 4 and 5). Although both designs have approximately equal low-frequency gain, the sensitive design rolls off much more quickly. The robust design has a higher loop gain in the region of uncertain frequencies (phase stabilization rather than gain stabilization). These results suggest that designs that attempt a high degree of control of the rigid-body mode relative to flexible modes may be sensitive to frequency uncertainty. The results also show that weighting strain energy in the cost functional does result in a closed-loop system that is more robust with respect to uncertainty in modal frequencies as predicted by the theory developed in previous sections.

V. Conclusions

Standard LQG/LTR methods are an effective way to achieve robust controller designs when the modeling errors of the plant are well characterized by a single unstructured uncertainty model. However, in the case of a flexible structure with uncertain frequencies, the unstructured uncertainty model is overly conservative. In this case a modification of the LQG procedure can be applied. The approach taken here sets up the control design procedure in terms of structured uncertainties, and then minimizes the 2-norm of the resulting transfer-function matrix. Once measurement noise is added and performance and control cost penalties are approximately adjusted, a well-posed LQG problem is obtained that can be solved with standard numerical methods. The results of Sec. IV demonstrate that this approach provides a significant improvement over standard LQG/LTR methods. As indicated in Sec. IV, robustness and performance can be traded off by adjusting two parameters until a suitable compromise is found.

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